A cry of distress from Nature? Fine tuning in scientific theories

Andrew Fowlie

Xi'an Jiaotong-Liverpool University

andrew.fowlie@xjtlu.edu.cn

andrewfowlie.github.io

School of CHIPS

Wisdom Forum

9 October 2024

Partially based on work with Gonzalo Herrera [2406.03533]

Overview

1. Fine tuning & physics beyond the Standard Model
2. Foundations
    - Frequentist methods
    - Bayesian
3. Recent developments in fine-tuning
4. Example

Section 1

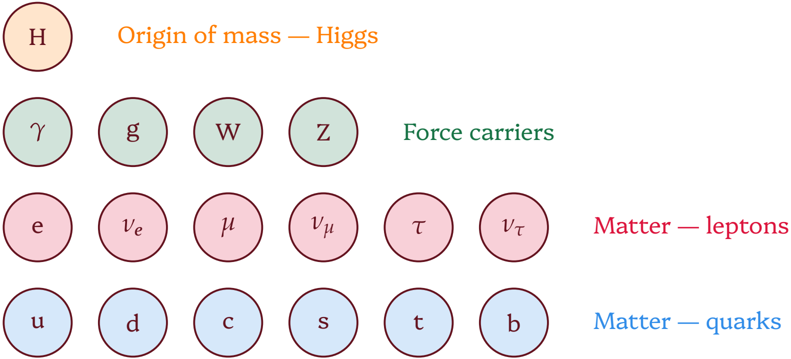
Fine tuning & physics beyond the Standard Model

The Standard Model

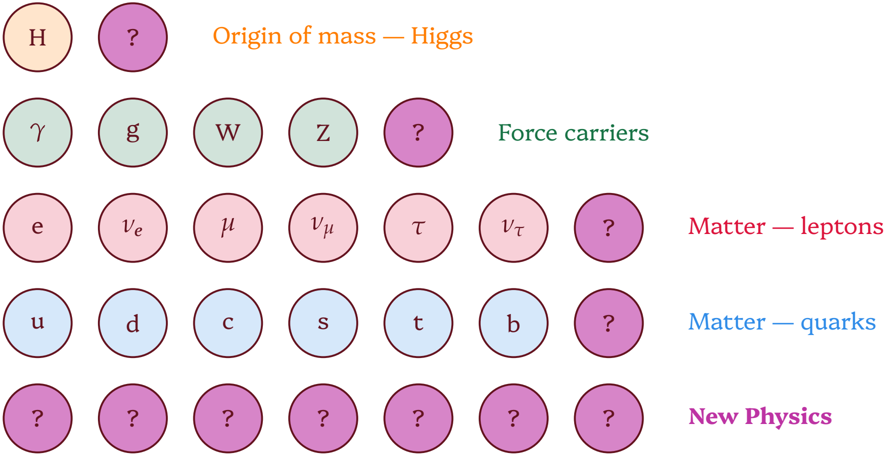
- The Standard Model is a theory in particle physics
- Describes all known particles and three interactions — the electromagnetic, weak, and strong nuclear interactions
- Remarkably successful in explaining and predicting the behavior of subatomic particles — exquisite agreement with experimental data
- Cannot be the whole story — what lies beyond the Standard Model?

Theorists and experimentalists are searching for clues

The Standard Model



Beyond the Standard Model



Experimental discoveries

Classic example. Higgs discovery in 2012.

How do we judge when the data indicates the presence of a new particle or phenomena?

Theoretical hints

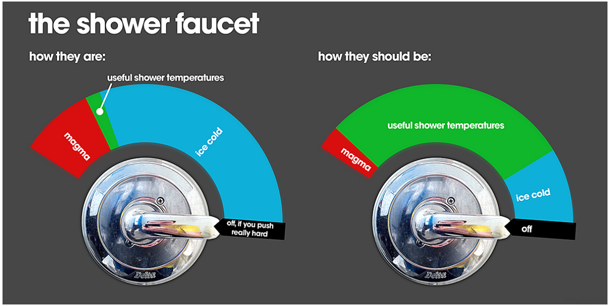
There are theoretical motivations for new particles and phenomena, e.g.,

- The hierarchy problem
- The strong CP problem
- Cosmological constant problem
- Horizon problem
- Flatness problem
- Grand Unification

These hints are motivated by ideas about **fine tuning.**

How do we judge whether these arguments are reliable?

Fine-tuning in everyday life

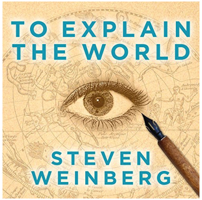


We know that showers that require **fine-tuning** are bad showers!

Fine-tuning in physics

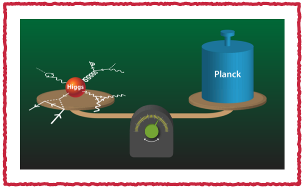
In high-energy physics, a theory is considered **fine-tuned or unnatural** if small variations in its parameters result in dramatic changes in its predictions. For reviews, see ref. [1–3]

> ***"Fine-tuning in a scientific theory is like a cry of distress from nature**, complaining that something needs to be better explained"* [4]



Hierarchy problem

The Standard Model Higgs mass parameter must be **fine tuned.** This is the hierarchy problem [5–9]



Requires **fine-tuning** of bare mass, ${m_0}^2$, and quantum corrections M^2

Hierarchy problem

The Standard Model Higgs mass parameter must be **fine tuned.** This is the hierarchy problem [5–9]

$$\boxed{\begin{aligned} m^2 &\simeq {m_0}^2 + M^2 \\ 100^2 &\simeq m_0^2 + (10^{19})^2 \end{aligned}}$$

Requires **fine-tuning** of bare mass, ${m_0}^2$, and quantum corrections M^2

$$m_0^2 = -99\,999\,999\,999\,999\,999\,999$$
$$999\,999\,999\,999\,990\,000 \text{ GeV}^2$$

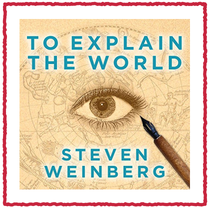
Solutions

- Since the 1980s, model-building in high-energy physics focussed on solving the hierachy problem
- In other words, building theories that didn't need fine-tuning
- All attempts to do so introduce new particles with masses just above 100 GeV
- **New physics that could be observed in particle colliders**
- Most popular models were supersymmetry (SUSY), including supersymmetric grand unified theories (GUTs)

Cosmological constant

There is a similar problem with a so-called cosmological constant, ρ, in cosmology.

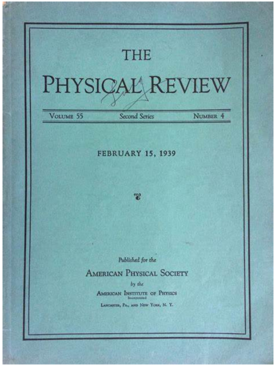
This parameter requires **fine-tuning** so that $\rho \lesssim 10^{-121}$ but corrections from known physics are at least 60 orders of magnitude greater [10]



*"**This level of fine-tuning is intolerable**, and theorists have been working hard to find a better way to explain why the amount of dark energy is so much smaller than that suggested by our calculations"* [4]

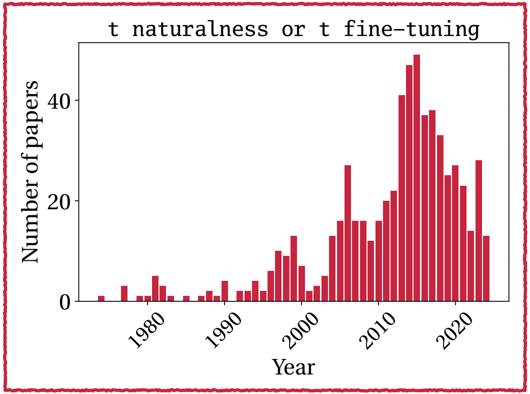
History of fine-tuning

- **1934** — Weisskopf's calculation of electron self-energy [11]
- **1938** — Dirac's large numbers hypothesis [12]
- **1973** — Wilson understanding of effective field theory [13]
- **1974** — Gaillard and Lee predict charm quark mass [14]
- **1988** — Weinberg makes anthropic argument [15]



Popularity of fine-tuning — data from INSPIRE

- **1974** — first hit by Georgi [16]
- **1979** — ’t Hooft [17]
- **1987** — Barbieri-Giudice measure [18]
- **2000** — fine-tuning at LEP [19]
- **2006** — pre-LHC forecasts
- **2010 onward** — LHC-era



Measures of fine-tuning

Fine-tuning of electroweak scale usually quantified by sensitivity measure [18, 20]

$$\boxed{\text{Sensitivity} = \frac{\text{Change in output}}{\text{Change in input}}}$$

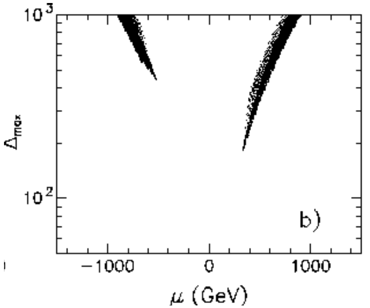
E.g. Barbieri-Giudice (BG)

$$\boxed{\Delta_{\text{BG}} = \left|\frac{\mathrm{d}\ln M_Z}{\mathrm{d}\ln a_i}\right| = \left|\frac{a_i}{M_Z}\frac{\mathrm{d}M_Z}{\mathrm{d}a_i}\right|}$$

What's the connection between these measures and plausible models?

Fine-tuning at LEP

Fine-tuning price of LEP [19, 21–23] — allowed points show $\Delta_{\mathrm{BG}} \gtrsim 100$




arXiv > hep-ph > arXiv:hep-ph/9712234

High Energy Physics - Phenomenology

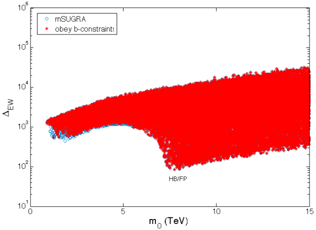
[Submitted on 2 Dec 1997 (v1), last revised 17 Dec 1997 (this version, v2)]

The fine-tuning price of LEP

P.H. Chankowski, J. Ellis, S. Pokorski

Fine-tuning at the LHC

Fine-tuning price of the LHC [24, 25] — allowed points show $\Delta_{BG} \gtrsim 1000$, except in focus-point region



There are, thus, now criticisms and doubts about fine-tuning [26]

Section 2

Foundations

Let the data speak for itself

"inanimate data can never speak for themselves, and we always bring to bear some conceptual framework, either intuitive and ill-formed, or tightly-formed and structured, to the task of investigation, analysis and interpretation" [27]

"No body of data tells us all we need to know about its own analysis" [28]

"The data cannot speak for themselves; and they never have, in any real problem of inference" [29]

Methodology

We need a statistical methodology to judge evidence. In the time available, let's consider

1. Frequentist; see e.g., [30–33]. Two schools
   - Error control
   - Evidential
2. Bayesian; see e.g., [34–39]

Testing and estimation

Roughly speaking, statistical tasks separate into

1. Model testing or comparison
2. Estimating or inferring the model's parameters

I will focus on first. In my opinion, first we should establish whether a phenomena exists, and then infer its parameters or properties.

Testing

Jeffreys and Fisher agree!

> *"[I]n what circumstances do observations support a change of the form of the law itself? This question is really logically prior to the estimation of the parameters, since the estimation problem presupposes that the parameters are relevant"* [40]

> *"It is a useful preliminary before making a statistical estimate ...to test if there is anything to justify estimation at all"* [41]

Likelihood

Methods typically require at least the **likelihood** (see e.g., [42])

$$L(\theta) = p(D \mid M, \theta)$$

This tells us the probability (density) of the **observed data**, D, given a particular model, M, and choice of parameters.

This is a function of the model's parameters, θ, for fixed, observed data.

P-values

P-value [43]

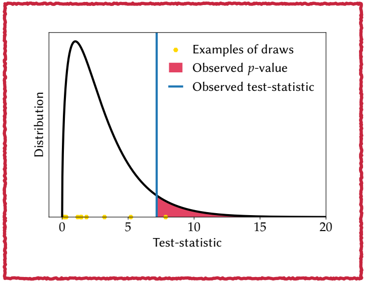
The p-value, p, is the probability of observing data as or more extreme than that observed, given the null hypothesis, H_0, i.e.,

$$p = P(\lambda \geq \lambda_{\text{Observed}} \mid H_0)$$

where λ is a test-statistic that summarises the data and defines extremeness, and H_0 specifies the distribution of λ

P-values

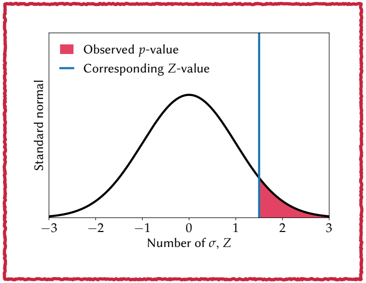
Thus p is a tail probability.



Thus p is uniformly distributed under H_0 (or dominated by uniform in discrete settings or composite null)

Z-values

In particle physics, it's common to translate p-values into Z-values. 5σ corresponds to about $p = 10^{-7}$. This is just a convention



through the equation $Z = \Phi^{-1}(1 - p)$

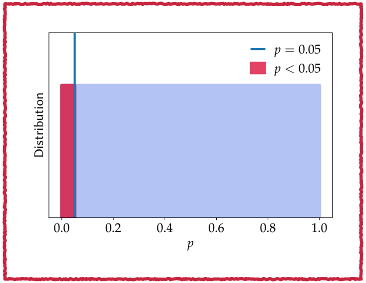
Interpreting p-values

P-values are popular in particle physics and elsewhere. Two possibly contradictory interpretations [44]:

- P is a **measure of evidence** against H_0 [41]: small $p \Rightarrow H_0$ implausible. See e.g., [45–49]
- P is a **means to control error rate** [50]: if we reject null when p-value ≤ 0.05, for example, becomes error theoretic approach with type 1 error rate $\alpha = 0.05$

Controlling type-1 error rate

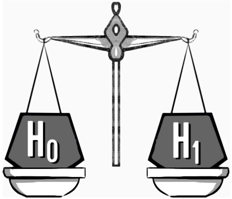
The p-value enables us to control type-1 error rate because it is uniformly distributed under the null



Placing a threshold $p < \alpha$ controls the type-one error rate to be α

Example from high-energy physics

Original artwork Viktor Beekman and concepts Eric-Jan Wagenmakers

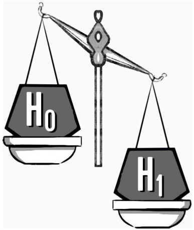


In high-energy physics, we want to discover new phenomena and new particles.

Perform null hypothesis test:

- H_0 — Standard Model (SM) backgrounds only
- H_1 — SM + new physics, e.g. Higgs boson or supersymmetric particles

Example from high-energy physics



For a discovery we conventionally require a tiny global p-value of

$$p \lesssim 10^{-7}\ (5\sigma)$$

i.e., $\alpha \simeq 10^{-7}$ [51]. Dual interpretation: threshold in evidence — extraordinary claims require extraordinary evidence — and imposes a 10^{-7} type-1 error rate.

No penalty for fine-tuning here

However we interpret it, **there is no penalty for fine-tuning in a *p*-value.**

The *p*-value conditions on the model,

$$\boxed{p = P(\lambda \geq \lambda_{\text{Observed}} \,|\, H_0)}$$

and doesn't care about whether that model was fine-tuned.

Let's try something else.

Bayesian inference

Forget long-run errors rates and data we don't have. Compute the change in plausibility of models in light of the data we have

$$p(A \mid B) = \frac{p(B \mid A)}{p(B)} \cdot p(A)$$

- We just apply probability theory to the problem [40]
- We could compute the relative change in plausibility of each model
- Simple in theory; in practice there are difficulties

Bayes factors

The Bayes factor [52] relates the relative plausibility of two models after data to their relative plausibility before data;

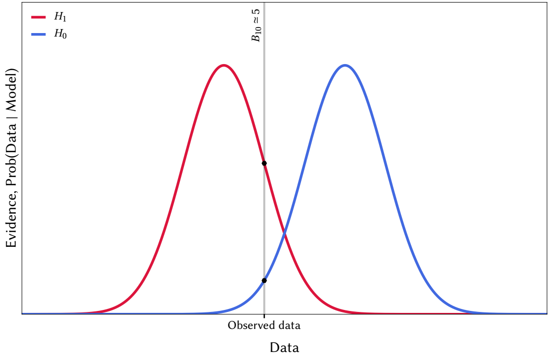
$$\boxed{\text{Posterior odds} = \textbf{Bayes factor} \times \text{Prior odds}}$$

where

$$\boxed{\text{Bayes factor} = \frac{p(\text{Observed data} \mid \text{Model } a)}{p(\text{Observed data} \mid \text{Model } b)}}$$

By applying laws of probability, we see that models should be compared by nothing other than **their ability to predict the observed data.**

Bayes factors



Bayesian evidence

The factors in the ratio are **Bayesian evidences**

$$\boxed{Z \equiv p(D \mid M) = \int L(\theta)\, \pi(\theta)\, \mathrm{d}\theta,}$$

where D is the observed data, $L(\theta) = p(D \mid \theta, M)$ is the likelihood and $\pi(\theta) = P(\theta \mid M)$ is our prior, and θ are the model's parameters.

The prior describes what we knew about the parameters before seeing the data

The evidence is the likelihood averaged over the prior — the averaging penalises fine-tuned models

Sensitivity to priors

Evidences are the likelihoods averaged over priors.

Many consider the resulting dependence of the Bayes factor on the priors to be a major and perhaps fatal problem; see e.g., [53, 54]

- **No priors, no predictions.** I need to compare your model's predictions with data. If you don't tell the plausible parameters, how am I to know what it predicts?
- **Sensitive to arbitrary choices.** If the inference changes dramatically within a class of reasonable priors, we can't draw reliable conclusions.

Sensitivity to priors

Evidences are the likelihoods averaged over priors.

Many consider the resulting dependence of the Bayes factor on the priors to be a major and perhaps fatal problem; see e.g., [53, 54]

> *"the lack of a concrete theory for choosing priors no more implies that one should not use Bayesian statistics than does the lack of a theory that tells us the right price to pay for groceries implies we should not use money"* [Paraphrasing Hill 1975]

Subjective & Objective

There are different approaches to constructing priors, leading to different flavors of Bayesian inference

Subjective

Priors reflect state of knowledge and could be constructed by e.g., consulting experts (see e.g., [55, 56])

Dictated by state of knowledge

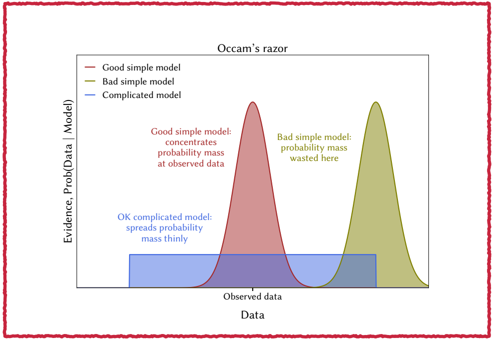
Priors could be dictated by e.g., a symmetry [57]

Formal rules for selecting priors

Construct priors that e.g., maximise what we expect to learn about a model's parameters [58, 59]

Occam's razor

The Bayesian evidence includes an automatic Occam's razor [60, 61]!



Could this justify fine-tuning arguments in physics?

Information theory

Bayesian inference is closely connected to information theory.

In particular, the Kullback-Leibler (KL) divergence between the prior and the posterior [62]

$$\boxed{D_{\mathrm{KL}} \equiv \int p(\theta \mid D) \ln\left[\frac{p(\theta \mid D)}{\pi(\theta)}\right] \mathrm{d}\theta}$$

is a measure of the **information learned about a parameter** [63]

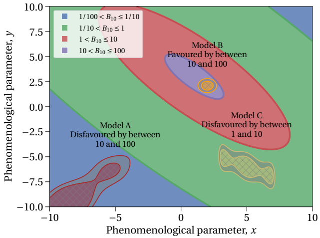
Section 3

Recent developments in fine-tuning

Statistical interpretation of fine-tuning

Theorist began to recognize that fine-tuning connected to probability of cancellations [64–69] and statistical inference [70–81]

Independently, we recently introduced the Bayes factor surface [82]



Statistical interpretation of fine-tuning

Theorist began to recognize that fine-tuning connected to probability of cancellations [64–69] and statistical inference [70–81]

Independently, we recently introduced the Bayes factor surface [82]

This shows the change in plausibility of a model as a function of that model's parameters relative to a reference model

$$\boxed{B(\theta) = \frac{p(D \,|\, M, \theta)}{p(D \,|\, M_0)}}$$

This is **a new way to understand the impact of experimental measurements**; see ref. [83–87] for recent related works in other contexts

Fine-tuning and Bayes factor surface

We found a link between the BG measure, statistics, and information theory. Consider the hierarchy problem and

- A model with parameters θ and ϕ that predicts Higgs mass parameter
- **Exchange ϕ for the measured Higgs mass parameter**
- Compare against a model with the Higgs mass parameter as an input parameter

$$\boxed{\begin{array}{c} B(\theta) = e^{\triangle D_{\text{KL}}} = \Delta_{\text{BG}} \\ \text{Bayes factor surface} = \text{Relative information} = \text{BG fine-tuning measure} \end{array}}$$

... for the parameter that was exchanged for the Higgs mass parameter

Interpretations of BG measure

- **Statistical** — the BG fine-tuning measure shows the Bayes factor surface versus an untuned model — *measures the change in plausibility of a model relative to an untuned model*

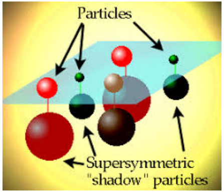
- **Information-theoretic** — the BG fine-tuning measure shows the compression versus an untuned model — *measures the exponential of the extra information, measured in nats, relative to an untuned model that you must supply about a parameter to fine-tune it*

Section 4

Example

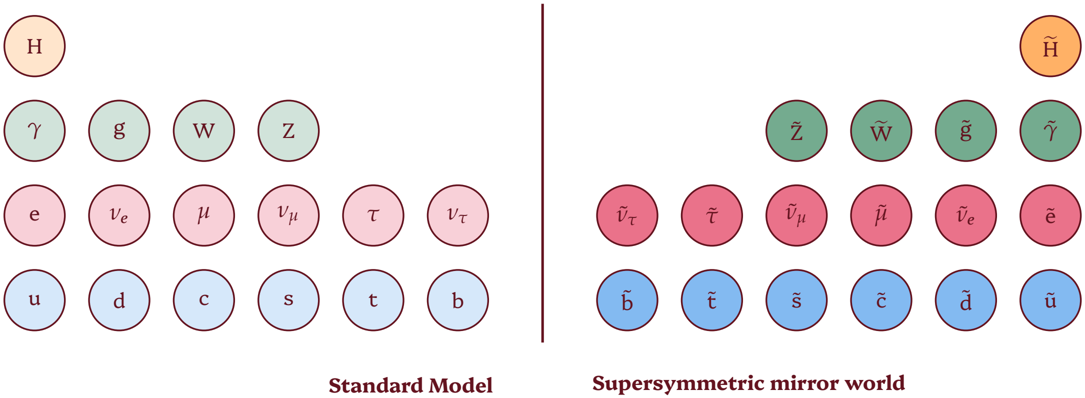
CMSSM/mSUGRA

This is a popular model based on symmetry called **supersymmetry**

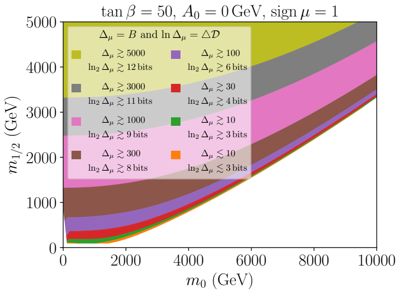


It introduces a supersymmetric mirror world. Cancellations between quantum corrections from the new particles alleivatiate the hierachy problem

Supersymmetric Mirror world



CMSSM/mSUGRA

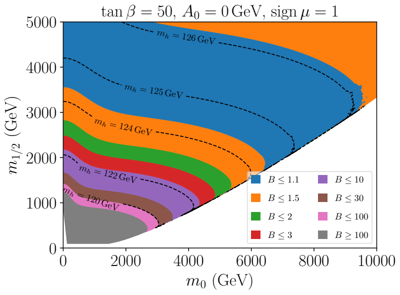


The traditional BG fine-tuning measure is equivalent to

- Bayes factor surface relative to untuned model — *CMSSM points disfavored by more than factor 300*
- Extra information that must be specified about a parameter — *at least 6 extra bits of information required about the μ-parameter*

... everywhere except in the narrow focus point strip where $\Delta_{BG} \leq 10$

CMSSM/mSUGRA

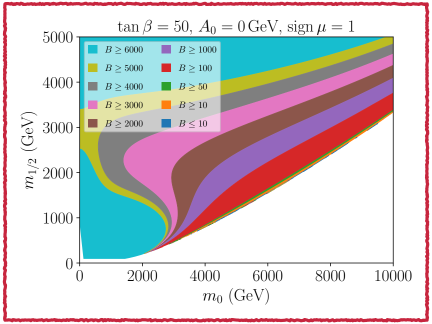


The Bayes factor surface for the $m_h \simeq 125\,\mathrm{GeV}$ Higgs mass measurement

- Computed relative to a reference model — *model that predicts $m_h = 125$ GeV with no tuning*
- Requires $m_0 \gg$ TeV and $m_{1/2} \gg$ TeV — *except in narrow focus-point*

How can we combine the Higgs mass measurement with the BG measure?

CMSSM/mSUGRA



Bayes factor surfaces from *Z* and Higgs mass measurements can be multipied

- The Z and Higgs mass measurements select narrow focus-point strip — *disfavoured, but only by* $B \leq 10$
- ... and rule out other choices — *disfavored by at least* $B > 100$

The BG measure should not be thought of as a χ^2*, but as a Bayes factor*

Conclusions

- Fine-tuning a “cry of distress from Nature” that motivates new physics
- Doubts raised about fine-tuning — arbitrariness, lack of logical foundation & negative results from LEP and LHC
- We found precise interpretations of the fine-tuning measure

  — **Statistical** — *measures the change in plausibility of a model relative to an untuned model*

  — **Information-theoretic** — *measures the extra information that you must supply about a parameter*

- Fine-tuning thus a legitimate principle and guide for new physics

References I

[1] P. Nelson, *Naturalness in theoretical physics: Internal constraints on theories, especially the requirement of naturalness, play a pivotal role in physics*, *Am. Sci.* **73** (1985) 60.

[2] G.F. Giudice, *Naturally Speaking: The Naturalness Criterion and Physics at the LHC*, in *Perspectives on LHC Physics*, G. Kane and A. Pierce, eds., pp. 155–178, World Scientific (2008), DOI [0801.2562].

[3] N. Craig, *Naturalness: past, present, and future*, *Eur. Phys. J. C* **83** (2023) 825 [2205.05708].

[4] S. Weinberg, *To Explain the World: The Discovery of Modern Science*, Penguin Books Limited (2015).

References II

[5] S. Weinberg, *Implications of Dynamical Symmetry Breaking*, *Phys. Rev.* **D13** (1976) 974.

[6] S. Weinberg, *Implications of Dynamical Symmetry Breaking: An Addendum*, *Phys. Rev.* **D19** (1979) 1277.

[7] L. Susskind, *Dynamics of Spontaneous Symmetry Breaking in the Weinberg-Salam Theory*, *Phys. Rev.* **D20** (1979) 2619.

[8] E. Gildener, *Gauge Symmetry Hierarchies*, *Phys. Rev.* **D14** (1976) 1667.

[9] E. Witten, *Dynamical Breaking of Supersymmetry*, *Nucl. Phys.* **B188** (1981) 513.

[10] R. Bousso, *TASI Lectures on the Cosmological Constant*, *Gen. Rel. Grav.* **40** (2008) 607 [`0708.4231`].

References III

[11] V. Weisskopf, *On the self-energy of the electron*, *Z. Phys.* **89** (1934) 27 [Erratum: Z.Phys. 90, 817–818 (1934)].

[12] P.A.M. Dirac, *New basis for cosmology*, *Proc. Roy. Soc. Lond. A* **165** (1938) 199.

[13] K.G. Wilson and J.B. Kogut, *The Renormalization group and the epsilon expansion*, *Phys. Rept.* **12** (1974) 75.

[14] M.K. Gaillard and B.W. Lee, *Rare Decay Modes of the K-Mesons in Gauge Theories*, *Phys. Rev. D* **10** (1974) 897.

[15] S. Weinberg, *The Cosmological Constant Problem*, *Rev. Mod. Phys.* **61** (1989) 1.

[16] H. Georgi and A. Pais, *Calculability and Naturalness in Gauge Theories*, *Phys. Rev. D* **10** (1974) 539.

References IV

[17] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, *NATO Sci. Ser. B* **59** (1980) 135 [iNSPIRE].

[18] R. Barbieri and G.F. Giudice, *Upper Bounds on Supersymmetric Particle Masses*, *Nucl. Phys. B* **306** (1988) 63 CERN-TH-4825/87.

[19] G.L. Kane and S.F. King, *Naturalness implications of LEP results*, *Phys. Lett. B* **451** (1999) 113 [hep-ph/9810374].

[20] J.R. Ellis, K. Enqvist, D.V. Nanopoulos and F. Zwirner, *Observables in Low-Energy Superstring Models*, *Mod. Phys. Lett. A* **1** (1986) 57 CERN-TH-4350-86.

[21] R. Barbieri and A. Strumia, *About the fine tuning price of LEP*, *Phys. Lett. B* **433** (1998) 63 [hep-ph/9801353].

References V

[22] P.H. Chankowski, J.R. Ellis and S. Pokorski, *The Fine tuning price of LEP*, *Phys. Lett. B* **423** (1998) 327 [hep-ph/9712234].

[23] P.H. Chankowski, J.R. Ellis, M. Olechowski and S. Pokorski, *Haggling over the fine tuning price of LEP*, *Nucl. Phys. B* **544** (1999) 39 [hep-ph/9808275].

[24] A. Strumia, *The Fine-tuning price of the early LHC*, *JHEP* **04** (2011) 073 [1101.2195].

[25] H. Baer, V. Barger, D. Mickelson and M. Padeffke-Kirkland, *SUSY models under siege: LHC constraints and electroweak fine-tuning*, *Phys. Rev. D* **89** (2014) 115019 [1404.2277].

[26] S. Hossenfelder, *Screams for explanation: finetuning and naturalness in the foundations of physics*, *Synthese* **198** (2021) 3727 [1801.02176].

References VI

[27] P. Gould, *Letting the Data Speak for Themselves*, *Annals of the Association of American Geographers* **71** (1981) 166.

[28] J.W. Tukey et al., *Exploratory data analysis*, vol. 2, Reading, MA (1977).

[29] E.T. Jaynes, *Probability theory: The logic of science*, Cambridge University Press (2003).

[30] L. Lyons, *Statistics for Nuclear and Particle Physicists*, Cambridge University Press (1989).

[31] G. Cowan, *Statistical Data Analysis*, Clarendon Press (1998).

[32] F. James, *Statistical Methods in Experimental Physics*, World Scientific (2006).

References VII

[33] O. Behnke, K. Kröninger, G. Schott and T. Schörner-Sadenius, *Data Analysis in High Energy Physics: A Practical Guide to Statistical Methods*, Wiley (2013).

[34] G. D'Agostini, *Bayesian Reasoning In Data Analysis: A Critical Introduction*, World Scientific Publishing Company (2003).

[35] P. Gregory, *Bayesian Logical Data Analysis for the Physical Sciences: A Comparative Approach with Mathematica® Support*, Cambridge University Press (2005).

[36] D. Sivia and J. Skilling, *Data Analysis: A Bayesian Tutorial*, Oxford University Press (2006).

[37] R. Trotta, *Bayes in the sky: Bayesian inference and model selection in cosmology*, *Contemp. Phys.* **49** (2008) 71 [0803.4089].

References VIII

[38] W. von der Linden, V. Dose and U. von Toussaint, *Bayesian Probability Theory: Applications in the Physical Sciences*, Cambridge University Press (2014).

[39] C. Bailer-Jones, *Practical Bayesian Inference: A Primer for Physical Scientists*, Cambridge University Press (2017).

[40] H. Jeffreys, *The Theory of Probability*, Oxford University Press (1939).

[41] R.A. Fisher, *Statistical Methods for Research Workers*, Oliver and Boyd (1925).

[42] R.D. Cousins, *What is the likelihood function, and how is it used in particle physics?*, *arXiv preprint* (2020) [2010.00356], CERN EP Newsletter.

[43] R.L. Wasserstein and N.A. Lazar, *The ASA Statement on p-Values: Context, Process, and Purpose*, *The American Statistician* **70** (2016) 129.

References IX

[44] R. Hubbard and M.J. Bayarri, *Confusion Over Measures of Evidence (p's) Versus Errors (α's) in Classical Statistical Testing*, *Am. Stat.* **57** (2003) 171.

[45] R. Hubbard and R.M. Lindsay, *Why P Values Are Not a Useful Measure of Evidence in Statistical Significance Testing*, *Theory & Psychology* **18** (2008) 69.

[46] M.J. Schervish, *P Values: What They are and What They are Not*, *Am. Stat.* **50** (1996) 203.

[47] J.O. Berger and T. Sellke, *Testing a Point Null Hypothesis: The Irreconcilability of P Values and Evidence*, *J. Am. Stat. Assoc.* **82** (1987) 112.

[48] S. Senn, *Two cheers for P-values?*, *Journal of Epidemiology and Biostatistics* **6** (2001) 193.

References X

[49] P.A. Murtaugh, *In defense of P values*, *Ecology* **95** (2014) 611.

[50] J. Neyman and E.S. Pearson, *On the Problem of the Most Efficient Tests of Statistical Hypotheses*, *Philos. Trans. Roy. Soc. London Ser. A* **231** (1933) 289.

[51] L. Lyons, *Discovering the Significance of 5 sigma*, *arXiv preprint* (2013) [`1310.1284`].

[52] R.E. Kass and A.E. Raftery, *Bayes Factors*, *J. Am. Statist. Assoc.* **90** (1995) 773.

[53] J.O. Berger and L.R. Pericchi, *Objective Bayesian methods for model selection: Introduction and comparison*, *IMS Lecture Notes – Monograph Series* **38** (2001) 135.

[54] R.D. Cousins, *Comment on 'Bayesian Analysis of Pentaquark Signals from CLAS Data', with Response to the Reply by Ireland and Protopopsecu*, *Phys. Rev. Lett.* **101** (2008) 029101 [`0807.1330`].

References XI

[55] M. Goldstein, *Subjective Bayesian Analysis: Principles and Practice*, *Bayesian Analysis* **1** (2006) .

[56] P. Mikkola, O.A. Martin, S. Chandramouli, M. Hartmann, O. Abril Pla et al., *Prior knowledge elicitation: The past, present, and future* [2112.01380].

[57] E.T. Jaynes, *Prior probabilities*, *IEEE Trans. Syst. Sci. Cybern.* **4** (1968) 227.

[58] R.E. Kass and L. Wasserman, *The Selection of Prior Distributions by Formal Rules*, *Journal of the American Statistical Association* **91** (1996) 1343.

[59] G. Consonni, D. Fouskakis, B. Liseo and I. Ntzoufras, *Prior Distributions for Objective Bayesian Analysis*, *Bayesian Analysis* **13** (2018) 627 .

References XII

[60] D.J. MacKay, *Bayesian methods for adaptive models*, Ph.D. thesis, 1992. 10.7907/H3A1-WM07.

[61] W.H. Jefferys and J.O. Berger, *Ockham's razor and Bayesian analysis*, *American Scientist* **80** (1992) 64.

[62] S. Kullback and R.A. Leibler, *On Information and Sufficiency*, *Ann. Math. Stat.* **22** (1951) 79.

[63] D.J. MacKay, *Information theory, inference and learning algorithms*, Cambridge University Press (2003).

[64] G.W. Anderson and D.J. Castano, *Measures of fine tuning*, *Phys. Lett. B* **347** (1995) 300 [hep-ph/9409419].

References XIII

[65] P. Ciafaloni and A. Strumia, *Naturalness upper bounds on gauge mediated soft terms*, *Nucl. Phys. B* **494** (1997) 41 [hep-ph/9611204].

[66] L. Giusti, A. Romanino and A. Strumia, *Natural ranges of supersymmetric signals*, *Nucl. Phys. B* **550** (1999) 3 [hep-ph/9811386].

[67] A. Strumia, *Naturalness of supersymmetric models* in *34th Rencontres de Moriond: Electroweak Interactions and Unified Theories*, pp. 441–446, 1999 [hep-ph/9904247].

[68] B.C. Allanach, *Naturalness priors and fits to the constrained minimal supersymmetric standard model*, *Phys. Lett. B* **635** (2006) 123 [hep-ph/0601089].

[69] P. Athron and D.J. Miller, *A New Measure of Fine Tuning*, *Phys. Rev. D* **76** (2007) 075010 [0705.2241].

References XIV

[70] B.C. Allanach, K. Cranmer, C.G. Lester and A.M. Weber, *Natural priors, CMSSM fits and LHC weather forecasts*, *JHEP* **08** (2007) 023 [0705.0487].

[71] M.E. Cabrera, J.A. Casas and R. Ruiz de Austri, *Bayesian approach and Naturalness in MSSM analyses for the LHC*, *JHEP* **03** (2009) 075 [0812.0536].

[72] M.E. Cabrera, *Bayesian Study and Naturalness in MSSM Forecast for the LHC* in *45th Rencontres de Moriond on Electroweak Interactions and Unified Theories*, pp. 517–520, May, 2010, https://inspirehep.net/record/855642/files/arXiv:1005.2525.pdf [1005.2525].

[73] C. Balazs, A. Buckley, D. Carter, B. Farmer and M. White, *Should we still believe in constrained supersymmetry?*, *Eur. Phys. J. C* **73** (2013) 2563 [1205.1568].

References XV

[74] S. Fichet, *Quantified naturalness from Bayesian statistics*, *Phys. Rev. D* **86** (2012) 125029 [1204.4940].

[75] A. Fowlie, *Is the CNMSSM more credible than the CMSSM?*, *Eur. Phys. J. C* **74** (2014) 3105 [1407.7534].

[76] A. Fowlie, *CMSSM, naturalness and the "fine-tuning price" of the Very Large Hadron Collider*, *Phys. Rev. D* **90** (2014) 015010 [1403.3407].

[77] A. Fowlie, *The little-hierarchy problem is a little problem: understanding the difference between the big- and little-hierarchy problems with Bayesian probability* [1506.03786].

[78] J.D. Clarke and P. Cox, *Naturalness made easy: two-loop naturalness bounds on minimal SM extensions*, *JHEP* **02** (2017) 129 [1607.07446].

References XVI

[79] A. Fowlie, C. Balazs, G. White, L. Marzola and M. Raidal, *Naturalness of the relaxion mechanism*, *JHEP* **08** (2016) 100 [1602.03889].

[80] P. Athron, C. Balazs, B. Farmer, A. Fowlie, D. Harries and D. Kim, *Bayesian analysis and naturalness of (Next-to-)Minimal Supersymmetric Models*, *JHEP* **10** (2017) 160 [1709.07895].

[81] P. Fundira and A. Purves, *Bayesian naturalness, simplicity, and testability applied to the BL MSSM GUT*, *Int. J. Mod. Phys. A* **33** (2018) 1841004 [1708.07835].

[82] A. Fowlie, *The Bayes factor surface for searches for new physics*, *Eur. Phys. J. C* **84** (2024) 426 [2401.11710].

References XVII

[83] C.T. Franck and R.B. Gramacy, *Assessing Bayes Factor Surfaces Using Interactive Visualization and Computer Surrogate Modeling*, *Am. Stat.* **74** (2020) 359 [1809.05580].

[84] V.E. Johnson, S. Pramanik and R. Shudde, *Bayes factor functions for reporting outcomes of hypothesis tests*, *Proc. Natl. Acad. Sci.* **120** (2023) [2210.00049].

[85] E.-J. Wagenmakers, Q.F. Gronau, F. Dablander and A. Etz, *The support interval*, *Erkenntnis* **87** (2020) 589–601 psyarxiv/zwnxb.

[86] S. Pawel, A. Ly and E.-J. Wagenmakers, *Evidential calibration of confidence intervals*, *Am. Stat.* **78** (2023) 1–11 [2206.12290].

References XVIII

[87] NANOGrav, *The NANOGrav 15 yr Data Set: Search for Signals from New Physics*, *Astrophys. J. Lett.* **951** (2023) L11 [2306.16219].